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Eigenvalue Bounds for Schrödinger Operators with a Homogeneous Magnetic Field

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Abstract. We prove Lieb-Thirring inequalities for Schrödinger operators with a homogeneous magnetic field in two and three space dimensions. The inequalities bound sums of eigenvalues by a semi-classical approximation which depends on the strength of the magnetic field, and hence quantifies the diamagnetic behavior of the system. For a harmonic oscillator in a homogenous magnetic field, we obtain the sharp constants in the inequalities.

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1. Introduction and Main Result

Lieb-Thirring inequalities [15] provide bounds on the sum of negative eigenvalues of Schrödinger operators in terms of a phase space integral. In this paper, we are interested in two-dimensional Schrödinger operators $H_B + V$ with a homogenous magnetic field of strength $B > 0$. Here

$$H_B = \left(-i\frac{\partial}{\partial x_1} + \frac{Bx_2}{2}\right)^2 + \left(-i\frac{\partial}{\partial x_2} - \frac{Bx_1}{2}\right)^2$$

is the Landau Hamiltonian in $L^2(\mathbb{R}^2)$ and V is a real-valued function. The Lieb-Thirring inequality states that

$$\operatorname{Tr}(H_B + V)_- \leq r_2\ (2\pi)^{-2} \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} \left(|p|^2 + V(x)\right)_- \,\mathrm{d}x\,\mathrm{d}p \tag{1}$$

with the (currently best, but presumably non-optimal) constant $r_2 = \pi/\sqrt{3}$ from [3]. Physically, the left side is (minus) the energy of a system of non-interacting fermions in an external potential V and an external, homogeneous magnetic field of strength B, whereas the right side is $-r_2$ times a semi-classical approximation to that energy.

Physically, one expects the system to show a diamagnetic behavior, that is, to have a higher energy in the presence of a magnetic field. This is however not reflected in (1), which has a right hand side independent of B. We refer to [6] for further references and a survey over this problem. Our goal in this letter is to obtain a bound similar to (1), but with a more refined semi-classical approximation which takes B into account. The approximation we propose is:

$$\frac{B}{2\pi}\sum_{m=0}^{\infty}\int_{\mathbb{R}^2}((2m+1)B+V(x))_-\,\mathrm{d}x. \tag{2}$$

This quantity reflects the diamagnetic behavior since,

$$\frac{B}{2\pi}\sum_{m=0}^{\infty}\int_{\mathbb{R}^2}((2m+1)B+V(x))_-\,\mathrm{d}x \le \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(|p|^2+V(x))_-\,\mathrm{d}x\,\mathrm{d}p \tag{3}$$

for every V. Inequality (3) follows (even before the x-integration) from an easy convexity inequality (see Lemma 12 below). We also note that when $B\to 0$, by a Riemann sum argument, the quantity (2) approaches

$$(4\pi)^{-1}\int_0^{\infty}\mathrm{d}E\int_{\mathbb{R}^2}(E+V(x))_-\,\mathrm{d}x = (2\pi)^{-2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(|p|^2+V(x)\right)_-\,\mathrm{d}x\,\mathrm{d}p, \tag{4}$$

which is the 'usual' phase space integral.

While the right side of (1) (up to the constant r_2) has the correct limiting behavior when a small parameter $\hbar$ is introduced, it is not useful in the coupled limit $B\to\infty$ and $\hbar\to 0$. This limit is physically relevant, for instance, in the study of neutron stars [13]. The magnetic quantity (2) reproduces the correct behavior in this regime. It is remarkable that this asymptotic profile is, indeed, a uniform, non-asymptotic bound. This is implicitly contained in [14] who use, however, only an approximation of (2). Our first result is:

THEOREM 1. *For any $B>0$ and any V on $\mathbb{R}^2$ one has,*

$$\operatorname{Tr}(H_B+V)_- \le \rho_2\,\frac{B}{2\pi}\sum_{m=0}^{\infty}\int_{\mathbb{R}^2}((2m+1)B+V(x))_-\,\mathrm{d}x \tag{5}$$

with $\rho_2 = 3$.

Hence, up to the moderate increase from $r_2 = \pi/\sqrt{3} \approx 1.81$ to $\rho_2 = 3$, we have found a magnetic analogue of (1) which reflects the desired diamagnetic behavior (3). An important ingredient in our proof is a method developed recently by Rumin [16] to derive kinetic energy inequalities; see Section 2.1.

Similarly as in the non-magnetic case, one might ask for the optimal value of the constant ρ_2. By the semi-classical result mentioned above one necessarily has $\rho_2 \geq 1$. A first result in this direction was obtained in [7] (extending previous work of [4]), where it was shown that if one takes V to be constant on a set of finite measure and plus infinity otherwise, then (5) holds with $\rho_2 = 1$. Our second main result is an analogous optimal bound for a harmonic oscillator.

THEOREM 2. *For any* $B > 0$, $\omega_1 > 0$, $\omega_2 > 0$ *and* $\mu > 0$, *inequality* (5) *holds with* $\rho_2 = 1$ *for* $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu$.

In particular, letting $B \to 0$ and using the limit in (4) we recover the known bounds in the non-magnetic case from [2,11]. Even though the eigenvalues of a harmonic oscillator in a homogeneous magnetic field are explicitly known (Lemma 10), the proof of Theorem 2 relies on a delicate property of a subclass of convex functions (Lemma 14) which, we feel, could be useful even beyond the context of this paper.

Moments of eigenvalues. Using some by now standard techniques we derive a few consequences of Theorems 1 and 2. First, following Aizenman and Lieb [1] one can replace V by $V - \mu$ in (5) and integrate with respect to μ to obtain that for any $\gamma \geq 1$

$$\operatorname{Tr}(H_B + V)_-^\gamma \leq \rho_2 \, \frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))_-^\gamma \, \mathrm{d}x, \tag{6}$$

where $\rho_2 = 3$ for general V and $\rho_2 = 1$ for $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu$. The restriction $\gamma \geq 1$ is necessary, since one easily checks that for $0 \leq \gamma < 1$ there is no constant ρ_2 such that (6) holds for all potentials V. Restricting ourselves to the quadratic case we shall show in Section 3.4

PROPOSITION 3. *For any* $0 \leq \gamma < 1$ *there are* $B > 0$, $\mu > 0$ *and* $\omega_1 = \omega_2$ *such that for* $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu$ *one has,*

$$\operatorname{Tr}(H_B + V)_-^\gamma > \frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))_-^\gamma \, \mathrm{d}x. \tag{7}$$

In particular, this shows that (6) does *not* hold with $\rho_2 = 1$ if $0 \leq \gamma < 1$, not even for a quadratic potential. Our counterexample in Proposition 3 appears in the limit

$\omega_j/B \to 0$ (with $\mu/B = 3$ fixed). Another counterexample can be obtained in the $B \to 0$ limit from the (non-magnetic) counterexample of Helffer-Robert [8] and the fact that for potentials of the special form $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu$ one has

$$\frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))_-^{\gamma} \, \mathrm{d}x \le \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (|p|^2 + V(x))_-^{\gamma} \, \mathrm{d}x \, \mathrm{d}p$$

for *all* $\gamma \ge 0$ [and not only for $\gamma \ge 1$, as in (3)].

Three dimensions. Next, we shall show that our bounds for $d = 2$ can be applied to deduce analogous bounds for $d = 3$. This argument is in the spirit of the lifting argument from [10–12]. We denote by $\hat{H}_B = H_B - \frac{\partial^2}{\partial x_3^2}$ the Landau Hamiltonian in $L^2(\mathbb{R}^3)$.

COROLLARY 4. *For any $B > 0$ and any $\hat{V}$ on $\mathbb{R}^3$, one has*

$$\mathrm{Tr}(\hat{H}_B + \hat{V})_- \le \rho_3 \, \frac{B}{(2\pi)^2} \sum_{m=0}^{\infty} \iint_{\mathbb{R}^3 \times \mathbb{R}} ((2m+1)B + p_3^2 + \hat{V}(x))_- \, \mathrm{d}x \, \mathrm{d}p_3 \tag{8}$$

with $\rho_3 = \sqrt{3}\pi$.

Proof. From the operator-valued Lieb-Thirring inequality of [3] we know that

$$\mathrm{Tr}(\hat{H}_B + \hat{V})_- \le \frac{\pi}{\sqrt{3}} \iint_{\mathbb{R} \times \mathbb{R}} \mathrm{Tr}_{L^2(\mathbb{R}^2)} (H_B + p_3^2 + \hat{V}(\cdot, x_3))_- \frac{\mathrm{d}x_3 \, \mathrm{d}p_3}{2\pi}.$$

Inequality (8) is therefore a consequence of Theorem 1. □

For the harmonic oscillator we have:

COROLLARY 5. *For any $B > 0$, $\omega_1 > 0$, $\omega_2 > 0$, $\omega_3 > 0$ and $\mu > 0$, inequality* (8) *holds with $\rho_3 = 1$ for $\hat{V}(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 - \mu$.*

Proof. We denote by E_j the eigenvalues of the one-dimensional harmonic oscillator $H = -\frac{\partial^2}{\partial x_3^2} + \omega_3^2 x_3^2$. Then, since $\hat{H}_B + \hat{V} = (H_B + V) \otimes I + I \otimes H$ with $V(x_1, x_2) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2$, we have

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} (\hat{H}_B + \hat{V})_- = \sum_j \mathrm{Tr}_{L^2(\mathbb{R}^2)} (H_B + V + E_j - \mu)_-.$$

According to Theorem 2 (which trivially holds for $\mu \le 0$ as well), this is bounded from above by

$$\frac{B}{2\pi}\sum_j\sum_{m=0}^{\infty}\int_{\mathbb{R}^2}\big((2m+1)B+V(x_1,x_2)+E_j-\mu\big)_-\,\mathrm{d}x_1\,dx_2$$
$$=\frac{B}{2\pi}\sum_{m=0}^{\infty}\int_{\mathbb{R}^2}\mathrm{Tr}_{L^2(\mathbb{R})}\,(H+(2m+1)B+V(x_1,x_2)-\mu)_-\,\mathrm{d}x_1\,\mathrm{d}x_2.$$

Next, we shall use that H satisfies a Lieb-Thirring inequality with semi-classical constant [2,11], that is, for any $\Lambda\in\mathbb{R}$,

$$\mathrm{Tr}_{L^2(\mathbb{R})}\,(H-\Lambda)_-\le\frac{1}{2\pi}\iint_{\mathbb{R}\times\mathbb{R}}\left(p_3^2+\omega_3^2x_3^2-\Lambda\right)_-\,\mathrm{d}x_3\,\mathrm{d}p_3.$$

(This can also be seen from Lemma 12 and recalling the explicit form of the eigenvalues of H.) It follows that for every fixed (x_1,x_2)

$$\begin{aligned}&\mathrm{Tr}_{L^2(\mathbb{R})}(H+(2m+1)B+V(x_1,x_2)-\mu)_-\\&\quad\le\frac{1}{2\pi}\iint_{\mathbb{R}\times\mathbb{R}}(p_3^2+\omega_3^2x_3^2+(2m+1)B+V(x_1,x_2)-\mu)_-\,\mathrm{d}x_3\,\mathrm{d}p_3,\end{aligned}$$

which proves the claimed bound. □

Remark 6. The previous proof shows that (8) with $\rho_3=1$ is valid for more general potentials $\hat V(x)=V(x_1,x_2)+v(x_3)$, where $V(x_1,x_2)=\omega_1^2x_1^2+\omega_2^2x_2^2$ and where v is such that $\mathrm{Tr}_{L^2(\mathbb{R})}(-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}+v(x_3)-\Lambda)_-\le\frac{1}{2\pi}\iint_{\mathbb{R}\times\mathbb{R}}(p_3^2+v(x_3)-\Lambda)_-\,\mathrm{d}x_3\,\mathrm{d}p_3$ for all Λ.

A similar argument as in the proofs of Corollaries 4 and 5 (based on the operator-valued Lieb-Thirring inequalities of [9,12]) shows that for general $\hat V$ one has

$$\mathrm{Tr}(\hat H_B+\hat V)_-^{\gamma}\le\rho_{3,\gamma}\ \frac{B}{(2\pi)^2}\sum_{m=0}^{\infty}\iint_{\mathbb{R}^3\times\mathbb{R}}((2m+1)B+p_3^2+\hat V(x))_-^{\gamma}\,\mathrm{d}x\,\mathrm{d}p_3 \tag{9}$$

with $\rho_{3,\gamma}=6$ if $\gamma\ge 1/2$, with $\rho_{3,\gamma}=\pi\sqrt{3}$ if $\gamma\ge 1$ and with $\rho_{3,\gamma}=3$ if $\gamma\ge 3/2$. Moreover, in the special case of $\hat V(x)=\omega_1^2x_1^2+\omega_2^2x_2^2+\omega_3^2x_3^2-\mu$, (9) holds with $\rho_3=1$ for $\gamma\ge1$ and with $\rho_{3,\gamma}=2(\gamma/(\gamma+1))^{\gamma}$ for $0\le\gamma<1$. The latter follows from the fact [7] that

$$\mathrm{Tr}_{L^2(\mathbb{R})}\left(-\frac{\mathrm{d}^2}{\mathrm{d}x_3^2}+\omega_3^2x_3^2-\Lambda\right)_-^{\gamma}\le2\left(\frac{\gamma}{\gamma+1}\right)^{\gamma}\frac{1}{2\pi}\iint_{\mathbb{R}\times\mathbb{R}}(p_3^2+\omega_3^2x_3^2-\Lambda)_-^{\gamma}\,\mathrm{d}x_3\,\mathrm{d}p_3.$$

2. Proof of Theorem 1

2.1. A KINETIC ENERGY INEQUALITY

We define a piecewise affine function $j:[0,\infty)\to[0,\infty)$ by

$$j(\rho)=\frac{B^2}{2\pi}(L^2+(2L+1)r) \quad \text{if } \rho=\frac{B}{2\pi}(L+r),\ \ L\in\mathbb{N}_0,\ \ r\in[0,1).$$

We note that j is continuous, increasing and convex. One has $j(\rho)=B\rho$ if $\rho\le B/(2\pi)$ and $j(\rho)\sim 2\pi\rho^2$ if $\rho\gg B$. The connection between this function and the right side of (5) will become clearer in the next section.

THEOREM 7. *Let $0\le\gamma\le 1$ be a density matrix on $L^2(\mathbb{R}^2)$ with finite kinetic energy. Then*

$$\operatorname{Tr} H_B\gamma \ge 3\int_{\mathbb{R}^2} j(\rho_\gamma(x)/3)\,\mathrm{d}x,$$

where $\rho_\gamma(x)=\gamma(x,x)$.

It is easy to see that $3\,j(\rho/3)\ge(1/3)\,j(\rho)$ for all $\rho\ge 0$, and therefore we also have

$$\operatorname{Tr} H_B\gamma \ge (1/3)\int_{\mathbb{R}^2} j(\rho_\gamma(x))\,\mathrm{d}x.$$

Proof. The first part of our proof follows the method introduced by Rumin [16]. We define $j_R:[0,\infty)\to[0,\infty)$ by

$$j_R(\rho)=B\rho+2B\sum_{k=1}^{\infty}\left(\sqrt{\rho}-\sqrt{\frac{Bk}{2\pi}}\right)_+^2.$$

We note that j_R is differentiable and convex, $j_R(\rho)=B\rho$ if $\rho\le B/(2\pi)$ and $j_R(\rho)\sim 2\pi\rho^2/3$ if $\rho\gg B$. We shall first show that

$$\operatorname{Tr} H_B\gamma \ge \int_{\mathbb{R}^2} j_R(\rho_\gamma(x))\,\mathrm{d}x. \tag{10}$$

In the second part of our proof (see Lemma 8) we show that $j_R(\rho)\ge 3\ j(\rho/3)$ for all $\rho\ge 0$.

For the proof of (10) we write:

$$\operatorname{Tr} H_B\gamma=\int_0^\infty \operatorname{Tr}(P^E\gamma)\,\mathrm{d}E=\int_{\mathbb{R}^2}\int_0^\infty \rho_\gamma^E(x)\,\mathrm{d}E\,\mathrm{d}x, \tag{11}$$

where P^E is the spectral projection of H_B corresponding to the interval $[E,\infty)$ and where $\rho_\gamma^E(x) = (P^E \gamma P^E)(x,x)$. It is well known that,

$$(1-P^E)(x,x) = \frac{B}{2\pi}\#\{m\in\mathbb{N}_0:\ (2m+1)B<E\}.$$

The same clever use of the triangle inequality as in [16] leads to the pointwise lower bound:

$$\rho_\gamma^E(x) \geq \left(\sqrt{\rho_\gamma(x)} - \sqrt{\frac{B}{2\pi}\#\{m\in\mathbb{N}_0:\ (2m+1)B<E\}}\right)_+^2. \tag{12}$$

Indeed, if $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm, we have for any set Ω of finite measure

$$\begin{aligned}\sqrt{\int_\Omega \rho_\gamma(x)\,\mathrm{d}x} &= \|\gamma^{1/2}\chi_\Omega\|_2 \leq \|\gamma^{1/2}P_E\chi_\Omega\|_2 + \|\gamma^{1/2}(1-P_E)\chi_\Omega\|_2 \\ &\leq \|\gamma^{1/2}P_E\chi_\Omega\|_2 + \|(1-P_E)^{1/2}\chi_\Omega\|_2 \\ &= \sqrt{\int_\Omega \rho_\gamma^E(x)\,\mathrm{d}x} + \sqrt{|\Omega|\frac{B}{2\pi}\#\{m\in\mathbb{N}_0:\ (2m+1)B<E\}}.\end{aligned}$$

Choosing Ω to be a ball around any fixed point x we obtain, as the radius of this ball tends to zero, that

$$\sqrt{\rho_\gamma(x)} \leq \sqrt{\rho_\gamma^E(x)} + \sqrt{\frac{B}{2\pi}\#\{m\in\mathbb{N}_0:\ (2m+1)B<E\}}.$$

This implies the claimed bound (12).

Inserting bound (12) in (11) we obtain

$$\begin{aligned}\operatorname{Tr} H_B\gamma &\geq \int_{\mathbb{R}^2}\left(\int_0^B \rho_\gamma(x)\,\mathrm{d}E + \sum_{k=1}^\infty \int_{(2k-1)B}^{(2k+1)B}\left(\sqrt{\rho_\gamma(x)} - \sqrt{\frac{Bk}{2\pi}}\right)_+^2 \mathrm{d}E\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^2} j_R(\rho_\gamma(x))\,\mathrm{d}x.\end{aligned}$$

This completes the proof of (10) and also, by Lemma 8 below, the proof of the theorem. $\square$

LEMMA 8. *$j_R(\rho) \geq 3\ j(\rho/3)$ for all $\rho \geq 0$.*

Proof. We are going to prove that

$$j_R(3\rho) \geq 3\ j(\rho). \tag{13}$$

Note that this is an equality for $\rho \le B/(6\pi)$. Moreover, since the left side of (13) is convex and the right side linear for $\rho \le B/(2\pi)$, we conclude that (13) holds for all $\rho \le B/(2\pi)$.

Henceforth, we shall assume that $\rho \ge B/(2\pi)$ and we write $3\rho = (B/2\pi)(K+s)$ for some integer $K \ge 3$ and some $s \in [0,1)$. If $K = 3L + m$ with $L \in \mathbb{N}$ and $m \in \{0,1,2\}$, then the lemma says that

$$K + s + 2\sum_{k=1}^{K}\left(\sqrt{K+s} - \sqrt{k}\right)^2 \ge 3(L^2 + \tfrac{1}{3}(2L+1)(m+s)).$$

We expand the square on the left side and insert $L = (K-m)/3$ on the right side. This shows that the assertion is equivalent to

$$K + s + 2K(K+s) - 4\sqrt{K+s}\sum_{k=1}^{K}\sqrt{k} + K(K+1) \ge \tfrac{1}{3}K^2 + \tfrac{2}{3}Ks + s + R,$$

for $K \in \mathbb{N}$ and $s \in [0,1)$, where $R = -\frac{1}{3}m^2 - \frac{2}{3}ms + m$. Since the inequality has to be true for any $m \in \{0,1,2\}$, we can replace R by its maximum over these m (with fixed s), that is, by $(2/3)(1-s)$. Thus, (13) is equivalent to

$$4K^2 + (3+2s)K - 6\sqrt{K+s}\sum_{k=1}^{K}\sqrt{k} - 1 + s \ge 0. \tag{14}$$

By the concavity of the square root we have

$$\frac{\sqrt{k} + \sqrt{k+1}}{2} \le \int_k^{k+1}\sqrt{t}\,\mathrm{d}t.$$

Summing this from $k=1$ to $k=K-1$ we get

$$\sum_{k=1}^{K}\sqrt{k} \le \int_1^K \sqrt{t}\,\mathrm{d}t + \frac{1+\sqrt{K}}{2} = \frac{2K^{3/2}}{3} + \frac{K^{1/2}}{2} - \frac{1}{6}.$$

This shows that

$$\begin{aligned} &4K^2 + (3+2s)K - 6\sqrt{K+s}\sum_{k=1}^{K}\sqrt{k} \\ &\quad \ge 4K^2 + (3+2s)K - \sqrt{K(K+s)}(4K+3) + \sqrt{K+s} \\ &\quad = \frac{sK((4s-12)K-9)}{4K^2 + (3+2s)K + \sqrt{K(K+s)}(4K+3)} + \sqrt{K+s}. \end{aligned}$$

In the quotient on the right side we estimate the numerator from below by $-3sK(4K+3)$ and the denominator from below by $4K^2 + 3K + K(4K+3) =$

$2K(4K+3)$. Thus, the quotient is bounded from below by $-3s/2$, and since $K\geq 3$ we conclude that

$$4K^2+(3+2s)K-6\sqrt{K+s}\sum_{k=1}^{K}\sqrt{k}-1+s\geq\sqrt{K+s}-1-\frac{s}{2}>0.$$

This proves (14) and completes the proof of the lemma. □

2.2. PROOF OF THEOREM 1

In this section, we are going to deduce Theorem 1 from Theorem 7. We define

$$p(v):=-\frac{B}{2\pi}\sum_{m=0}^{\infty}((2m+1)B+v)_-$$

for $v\in\mathbb{R}$. This is a concave, increasing and non-positive function. The key observation is that this p is the Legendre transform of the function $-j$ from the previous subsection, that is,

$$p(v)=\inf_{\rho\geq 0}\left(j(\rho)+v\rho\right). \tag{15}$$

This can be verified by elementary computations.

In order to prove Theorem 1 we apply Theorem 7 to get the estimate

$$\operatorname{Tr}(H_B+V)\gamma\geq\int_{\mathbb{R}^2}\big(3j(\rho_\gamma(x)/3)+V(x)\rho_\gamma(x)\big)\,\mathrm{d}x$$

for any $0\leq\gamma\leq 1$. According to (15) this is bounded from below by $3\int_{\mathbb{R}^2}p(V(x))\,\mathrm{d}x$. For γ equal to the projection corresponding to the negative spectrum of H_B+V we obtain the assertion of Theorem 1.

Remark 9. Similar arguments show that Theorem 7 can be deduced from Theorem 1. Indeed, since j is convex it is its double Legendre transform. By (15) we obtain

$$j(\rho)=\sup_{v\in\mathbb{R}}\left(p(v)-v\rho\right). \tag{16}$$

By the variational principle and Theorem 1 we can estimate for any $0\leq\gamma\leq 1$ and any V

$$\operatorname{Tr}H_B\gamma\geq-\operatorname{Tr}(H_B+V)_--\int_{\mathbb{R}^2}V(x)\rho_\gamma(x)\,\mathrm{d}x\geq\int_{\mathbb{R}^2}\big(3p(V(x))-V(x)\rho_\gamma(x)\big)\,\mathrm{d}x.$$

Because of (16) we can choose the function V such that the right side is equal to $3\int_{\mathbb{R}^2}j(\rho_\gamma(x)/3)\,\mathrm{d}x$, and this shows Theorem 7.

3. Proof of Theorem 2

3.1. THE SPECTRUM OF $H_B + V$

The explicit form of the eigenvalues of $H_B + \omega^2|x|^2$ was discovered in [5]. We include an alternative derivation of this result, which is also valid in the non-radial case.

LEMMA 10. *For any $B > 0$ and $\omega_1, \omega_2 > 0$ the operator $H_B + \omega_1^2 x_1^2 + \omega_2^2 x_2^2$ has discrete spectrum and its eigenvalues, including multiplicities, are given by*

$$B\left(a_+\left(\frac{\omega_1}{B}, \frac{\omega_2}{B}\right)(2k+1) + a_-\left(\frac{\omega_1}{B}, \frac{\omega_2}{B}\right)(2l+1)\right), \quad k, l \in \mathbb{N}_0,$$

where

$$a_\pm(\sigma_1, \sigma_2) = \sqrt{\frac{1}{2}\left(1 + \sigma_1^2 + \sigma_2^2 \pm \sqrt{(1+\sigma_1^2+\sigma_2^2)^2 - 4\sigma_1^2\sigma_2^2}\right)}. \tag{17}$$

Remark 11. It will be important for our analysis below that

$$a_-(\sigma)\, a_+(\sigma) = \sigma_1 \sigma_2, \tag{18}$$

which is easily checked.

Proof. By means of the gauge transform $e^{-iBx_1x_2/2}$ we see that $H_B + V$ is unitarily equivalent to the operator

$$-\frac{\partial^2}{\partial x_1^2} + \left(-i\frac{\partial}{\partial x_2} - Bx_1\right)^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2,$$

which, in turn, by a partial Fourier transform with respect to x_2, is unitarily equivalent to

$$-\frac{\partial^2}{\partial x_1^2} + (x_2 - Bx_1)^2 + \omega_1^2 x_1^2 - \omega_2^2 \frac{\partial^2}{\partial x_2^2}.$$

After scaling $x_2 \mapsto \omega_2 x_2$ this becomes the non-radial harmonic oscillator $-\Delta + x^t Ax$ with the matrix

$$A = \begin{pmatrix} B^2 + \omega_1^2 & -B\omega_2 \\ -B\omega_2 & \omega_2^2 \end{pmatrix}.$$

The eigenvalues of A are $B^2 a_+(\omega_1/B, \omega_2/B)^2$ and $B^2 a_-(\omega_1/B, \omega_2/B)^2$. Using the eigenvectors of A as basis in $\mathbb{R}^2$, we obtain a direct sum of two one-dimensional harmonic oscillators with frequencies Ba_+ and Ba_-, respectively. This leads to the stated form of the eigenvalues. $\square$

According to Lemma 10 and a simple computation, (5) with $\rho_2 = 1$ is equivalent to

$$\sum_{k,l\geq 0}\left(\mu - Ba_+\left(\frac{\omega_1}{B},\frac{\omega_2}{B}\right)(2k+1) - Ba_-\left(\frac{\omega_1}{B},\frac{\omega_2}{B}\right)(2l+1)\right)_+ \\ \leq \frac{B}{4\omega_1\omega_2}\sum_{m\geq 0}(\mu-(2m+1)B)_+^2$$

with $a_\pm$ given by (17). Setting $\Lambda = \mu/B$, $\sigma_j = \omega_j/B$ and $a_\pm = a_\pm(\sigma)$ and substituting (18) we can rewrite the desired inequality as

$$\sum_{k,l\geq 0}(\Lambda - a_+(2k+1) - a_-(2l+1))_+ \leq \frac{1}{4a_-a_+}\sum_{m\geq 0}(\Lambda-(2m+1))_+^2, \tag{19}$$

and this is what we shall prove.

3.2. TWO INEQUALITIES FOR CONVEX FUNCTIONS

For the proof of (19) we shall need

LEMMA 12. *Let ϕ be a non-negative convex function on $(0,\infty)$ such that $\int_0^\infty \phi(t)\,dt$ exists. Then*

$$\sum_{k=0}^{\infty}\phi(k+\tfrac{1}{2}) \leq \int_0^\infty \phi(t)\,\mathrm{d}t.$$

Proof. Indeed, by the mean-value property of convex functions $\phi(k+\frac{1}{2}) \leq \int_k^{k+1}\phi(t)\,\mathrm{d}t$ for each k. Now sum over k. □

Remark 13. The proof also shows that $\sum_{k=0}^{K-1}\phi(k+\frac{1}{2}) \leq \int_0^K \phi(t)\,\mathrm{d}t$ for each integer K. This observation will be useful later.

The inequality from Lemma 12 is sufficient to prove a sharp Lieb-Thirring inequality in the non-magnetic case, but for the proof of our Theorem 2 we need a more subtle fact about convex functions. We note that by the previous lemma $h\sum_{k=0}^{\infty}\phi(h(k+\frac{1}{2})) \leq \int_0^\infty \phi(t)\,\mathrm{d}t$ for any $h > 0$. Moreover, $h\sum_{k=0}^{\infty}\phi(h(k+\frac{1}{2})) \to \int_0^\infty \phi(t)\,\mathrm{d}t$ as $h \to 0$ by the definition of the Riemann integral. The key for proving our sharp result is that, for a certain subclass of convex functions, this limit is approached monotonically. More precisely, one has

LEMMA 14. *Let ϕ be a non-negative convex function on $(0,\infty)$ such that $\int_0^\infty \phi(t)\mathrm{d}t$ exists. Assume that ϕ is differentiable and that ϕ' is concave. Then the sum*

$$h\sum_{k=0}^{\infty}\phi\left(h\left(k+\frac{1}{2}\right)\right)$$

is decreasing in the parameter $h>0$.

We emphasize that without assumptions on ϕ' the inequality

$$\sum_{k=0}^{\infty}\phi\left(k+\frac{1}{2}\right)\leq h\sum_{k=0}^{\infty}\phi\left(h\left(k+\frac{1}{2}\right)\right)$$

is not true for all $h<1$. Indeed, take for instance $\phi(t)=(1-t)_+$ and $h\geq 2/3$.

In the proof of this lemma we shall make use of the following well-known fact about convex functions: If ψ is a non-negative convex function on $(0,\infty)$ such that $\int_0^\infty \psi(t)\,\mathrm{d}t$ exists, then $\psi(t)=\int_0^\infty (T-t)_+\,\mathrm{d}\mu(T)$ for some non-negative measure μ. Indeed, it is known that the left-sided derivative $\partial_-\psi$ exists everywhere on $(0,\infty)$ and satisfies $\psi(b)-\psi(a)=\int_a^b \partial_-\psi(t)\,\mathrm{d}t$ for $0<a<b<\infty$. Moreover, $\partial_-\psi$ is increasing and left-continuous, and therefore there is a non-negative measure μ such that $\partial_-\psi(b)-\partial_-\psi(a)=\mu([a,b))$. Since $\lim_{t\to\infty}\psi(t)=\lim_{t\to\infty}\partial_-\psi(t)=0$, we have by Fubini's theorem

$$\psi(t)=-\int_t^\infty \partial_-\psi(a)\,\mathrm{d}a=\int_t^\infty\left(\int \chi_{[a,\infty)}(T)\,\mathrm{d}\mu(T)\right)\mathrm{d}a=\int_0^\infty (T-t)_+\,\mathrm{d}\mu(T),$$

as claimed.

Proof. According to the fact recalled above (applied to $\psi=-\phi'$) we have $\phi(t)=\int_0^\infty (T-t)_+^2\,\mathrm{d}\mu(T)$ for a non-negative measure μ. Hence it suffices to prove the lemma for $\phi(t)=(T-t)_+^2$ with $T>0$. We have to prove that $\sum_{k=0}^\infty(\phi(h(k+\frac{1}{2}))+h(k+\frac{1}{2})\phi'(h(k+\frac{1}{2}))\leq 0$, which for our ϕ reads

$$\sum_{k=0}^{\infty}((S-2k-1)_+^2-2(2k+1)(S-2k-1)_+)\leq 0,$$

with $S=2T/h$. Choose $K\in\mathbb{N}_0$ such that $2K-1\leq S<2K+1$. Then the left side above equals

$$\begin{aligned}\sum_{k=0}^{K-1}((S-2k-1)^2-2(2k+1)(S-2k-1))&=\sum_{k=0}^{K-1}(S^2-4S(2k+1)+3(2k+1)^2)\\&=K(S^2-4SK+(2K-1)(2K+1))=K(S-2K+1)(S-2K-1).\end{aligned}$$

This is clearly non-positive for $2K-1\leq S<2K+1$, thus proving the claim. □

3.3. PROOF OF THEOREM 2

We have to prove (19). By Lemma 12 for any k

$$\sum_{l\geq 0}(\Lambda - a_+(2k+1) - a_-(2l+1))_+ \leq \int_0^\infty (\Lambda - a_+(2k+1) - 2a_- t)_+ \,\mathrm{d}t = \frac{1}{4a_-}(\Lambda - a_+(2k+1))_+^2 .$$

A simple computation shows that $a_+ = a_+(\sigma) \geq 1$, and hence by Lemma 14

$$a_+ \sum_{k\geq 0}(\Lambda - a_+(2k+1))_+^2 \leq \sum_{k\geq 0}(\Lambda - (2k+1))_+^2 .$$

The previous two inequalities imply the desired (19). □

3.4. PROOF OF PROPOSITION 3

Given $0 \leq \gamma < 1$, we want to find $\omega_1 = \omega_2$ and B such that the reverse inequality (7) holds. We may assume $\gamma > 0$ in the following. (The case $\gamma = 0$ can be treated similarly, or one may use the argument of Aizenman and Lieb mentioned in the introduction to conclude that a counterexample for $\gamma = \gamma_0$ implies one for all $\gamma < \gamma_0$.)

By the same computation that lead to (19) we see that (7) can be written as:

$$\sum_{k,l\geq 0}(\Lambda - a_+(2k+1) - a_-(2l+1))_+^\gamma > \frac{1}{2(\gamma+1)a_- a_+}\sum_{m\geq 0}(\Lambda - (2m+1))_+^{\gamma+1}$$

with $\Lambda = \mu/B$, $\sigma_j = \omega_j/B$ and $a_\pm = a_\pm(\sigma)$. We will let $\omega_1 = \omega_2$ and use the notation $t = \sigma^2$. One can show that $a_+ = 1 + t + O(t^2)$ and $a_- = t + O(t^2)$ as $t \to 0+$. We now choose $\Lambda = 3$ and recall that $a_+ = a_+(\sigma) \geq 1$. This gives us the inequality

$$2(\gamma+1)a_- a_+ \sum_{l\geq 0}(3 - a_+ - a_-(2l+1))_+^\gamma - 2^{\gamma+1} > 0,$$

which may be written as

$$(\gamma+1)a_-^{\gamma+1} a_+ \sum_{l\geq 0}(x-l)_+^\gamma - 1 > 0$$

with $x = (3 - a_+ - a_-)/(2a_-)$. Since $x = t^{-1}(1 + O(t))$ as $t \to 0+$, we may choose σ so that x is an integer. In this case we may use the concavity of y^γ and Remark 13 to bound

$$\sum_{l\geq 0}(x-l)_+^\gamma = \sum_{l=1}^{x} l^\gamma \geq \int_{1/2}^{x+1/2} t^\gamma \,\mathrm{d}t = \frac{1}{\gamma+1}((x+1/2)^{\gamma+1} - (1/2)^{\gamma+1}).$$

This shows that

$$
\begin{aligned}
(\gamma+1)a_-^{\gamma+1}a_+\sum_{l\geq 0}(x-l)_+^\gamma &\geq a_+((a_-x+a_-/2)^{\gamma+1}-(a_-/2)^{\gamma+1})\\
&= a_+(((3-a_+)/2)^{\gamma+1}-(a_-/2)^{\gamma+1})\\
&= (1+t+O(t^2))(1-t/2+O(t^2))^{\gamma+1}+O(t^{\gamma+1})\\
&= 1+\frac{1-\gamma}{2}t+O(t^{\gamma+1}).
\end{aligned}
$$

Since this is strictly larger than 1 for sufficiently small t, we have proved our claim. □

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